

Cosmological-Billiards Groups and self-adjoint BKL Transfer Operators

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Cosmological billiards arise as a map of the solution of the Einstein equations, when the most general symmetry for the metric tensor is hypothesized, and points are considered as spatially decoupled in the asymptotic limit towards the cosmological singularity, according to the BKL (Belinski Khalatnikov Lifshitz) paradigm. In $4 = 3 + 1$ dimensions, two kinds of cosmological billiards are considered: the so-called 'big billiard' which accounts for pure gravity, and the 'small billiard', which is a symmetry-reduced version of the previous one, and is obtained when the 'symmetry walls' are considered.

The solution of Einstein field equations is this way mapped to the (discrete) Poincaré map of a billiard ball on the sides of a triangular billiard table, in the Upper Poincaré Half Plane (UPHP).

The billiard modular group is the scheme within which the dynamics of classical chaotic systems on surfaces of constant negative curvature is analyzed. The periodic orbits of the two kinds of billiards are classified, according to the different symmetry-quotienting mechanisms.

The differences with the description implied by the billiard modular group are investigated and outlined.

In the quantum regime, the eigenvalues (i.e. the sign that wavefunctions acquire according to quantum BKL maps) for periodic phenomena of the BKL maps on the Maass wavefunctions are classified. The complete spectrum of the semiclassical operators which act as BKL map for periodic orbits is obtained.

Differently from the case of the modular group, here it is shown that the semiclassical transfer operator for Cosmological Billiards is not only the adjoint operator of the one acting on the Maass waveforms, but that the two operators are the same *self-adjoint operator*, thus outlining a different approach to the Langlands Jaquet correspondance.

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I. INTRODUCTION

The solution to the Einstein field equations in the asymptotic limit towards the cosmological singularity corresponds, within the BKL (named after Belinski, Khalatnikov and Lifshitz) Ref. 1, Ref. 2, Ref.3, Ref. 4, Ref.6 Ref. 7 paradigm, for which space gradients are neglected with respect to time derivatives, under the most general assumptions for the symmetries of the metric tensor, to the asymptotic limit towards the cosmological singularity of a Bianchi IX Universe. This way, the Einstein field equations for a Generic Universe correspond to a system of ordinary differential equations, whose symmetry for homogeneous universes and for inhomogeneous universe, $\Gamma_2(PGL(2, C))$ and c , respectively, allow one to define the corresponding billiard systems on the Upper Poincaré Half Plane, and the corresponding billiard maps for the discretized dynamics Ref.11, Ref.12, Ref. 13.

The conjugacy subclasses of the Billiard Modular Group define the $\Gamma_2(PGL(2, C))$ congruence subgroup of $PGL(2, C)$. The discretization of the dynamics allows for the definition of language codes, i.e. the composition of transformations that describe the continuous billiard dynamics and according to the time evolution of the Einstein field equations; the analyses in [44] therefore cannot be applied to Cosmological Billiards.

The unquotiented cosmological billiard groups therefore describe the (time) evolution of the dynamics, for which the insertion of any composition of transformations corresponding to the Identity within the unquotiented maps would correspond to (with respect to the time evolution and therefore with respect to the symmetries of the solution of the Einstein field equations) unphysical sequences of trajectories.

The eigenvalues for the BKL quantum operators that constitute the quantum maps and the semiclassical ones are defined according to the language codes of the big billiard and of the small billiard.

Cyclic identities for periodic orbits of the big billiard map and of the small billiard map define the parity of the (with respect to the corresponding WDW equation) suitable Maass wavefunctions and therefore exactly solve the Selberg trace formula for cosmological billiards.

The paper is organized as follows.

In Section II, the Billiard Modular Group is defined.

In Section III, the language code and the definition of periodic orbits for the Big Billiard Group are defined.

In Section IV, the language code and the definition of periodic orbits for the Small Billiard Group are defined.

In Section V, a comparison of the systems is outlined.

In Section VI, the quantum regime and the semiclassical transition are analyzed by the definition of the semiclassical Poincaré surface of section for cosmological billiards and the implementation of the Selberg trace formula according to the sign acquired by the eigenvalues of the quantum BKL maps for periodic Cosmological Billiards orbits.

Outlook and Perspectives are briefly stated in Section VII.

Brief concluding remarks follow.

II. THE BILLIARD MODULAR GROUP

The Billiard Modular Group (BMG) is defined on the asymmetric domain delimited by the geodesics \mathcal{A} , \mathcal{B} , \mathcal{C} , such that

$$\mathcal{A} : u = 0, \tag{2.1a}$$

$$\mathcal{B} : u = \frac{1}{2} \tag{2.1b}$$

$$\mathcal{C} : u^2 + v^2 = 1, \tag{2.1c}$$

for which the following transformations are defined

$$\mathcal{A}z = -\bar{z}, \tag{2.2a}$$

$$\mathcal{B}z = 1 - \bar{z}, \tag{2.2b}$$

$$\mathcal{C} = \frac{1}{z}. \tag{2.2c}$$

a. The language code for the BMG According to these transformation for the asymmetric domain (2.1), any matrix of the BMG can be written as one of the following

- $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{AC}, \mathcal{BC}, (\mathcal{BC})^2;$

- $I, \mathcal{B}AT^n, \mathcal{B}T^n, T^n\mathcal{A}, \mathcal{B}T^n\mathcal{A}, \mathcal{B}\mathcal{A};$
- $MCT^{n_1}CT^{n_2}\dots CT^{n_k}N,$

where the matrix I is the identity matrix, the matrix T is defined as $T = \mathcal{A}\mathcal{B}$, such that $Tz = z - 1$, T^n is its n -th iteration, the matrix M can be one of the following: $I, T^n, \mathcal{B}T^n, \mathcal{C}\mathcal{B}, \mathcal{B}$; the matrix N can be one of the following: $I, \mathcal{C}, \mathcal{A}, \mathcal{A}T^n, \mathcal{C}\mathcal{A}$. This way, the *language code* for the BMG is defined.

By a suitable transformation (conjugation), any matrix of the BMG can be written as one of the following

- elliptic transformations $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{AC}, \mathcal{BC}, (\mathcal{BC})^2;$
- parabolic transformations $T^n;$
- hyperbolic transformations $CT^{n_1}CT^{n_2}\dots CT^{n_k}.$

b. Action of the BMG on the oriented endpoints The periodic orbits of the MBG can be described by hyperbolic transformations by a sequence of integers n_1, n_2, \dots, n_k , after identification of circular permutations, for which reduced matrices are defined.

A periodic orbit of the BMG is defined by two oriented endpoints of a geodesics, $-1 < x' < 0$ and $x > 1$, which are invariant under the action of hyperbolic transformations, which define a quadratic equation with integer coefficients, i.e.

$$CT^{n_1}CT^{n_2}\dots CT^{n_k}x' = x', \quad (2.3a)$$

$$CT^{n_1}CT^{n_2}\dots CT^{n_k}x = x, \quad (2.3b)$$

such that the continued-fraction decomposition of x reads

$$x = n_1 + \frac{1}{n_2 + \frac{1}{\dots n_k + \frac{1}{n_1 + \dots}}}, \quad (2.4)$$

the continued-fraction decomposition of x' reads

$$-x' = n_1 + \frac{1}{n_2 + \frac{1}{\dots n_k + x'}}. \quad (2.5)$$

The code of matrices, the continued-fraction expansion of the oriented endpoints of the geodesics and the expression of the roots of the quadratic forms are equivalent for the definition of the conjugacy subclasses of the BMG, and can be used to gain information one from the others.

A. Comparison with the $SL(2, \mathbb{Z})$ group

The group $SL(2, \mathbb{Z})$ is defined on the symmetric fundamental domain delimited by the sides A_1, A_2, A_3 , described by

$$A_1 : u = -\frac{1}{2}, \quad (2.6a)$$

$$A_2 : u = \frac{1}{2}, \quad (2.6b)$$

$$A_3 : u^2 + v^2 = 1, \quad (2.6c)$$

with the transformations

$$T(z) = z + 1, \quad (2.7a)$$

$$S(z) = -\frac{1}{z} \quad (2.7b)$$

The sides are identified as

$$T : A_1 \rightarrow A_2, \quad (2.8a)$$

$$S : A_3 \rightarrow A_3. \quad (2.8b)$$

In particular, it is straightforward verified that the transformation S in Eq. (2.7b) acts in Eq. (2.8b) by identifying the $u > 0$ part of the side A_3 in Eq. (2.6c) with the $u < 0$ part of the same side, and viceversa.

The asymmetric domain of the BMG (2.1) is obtained by a suitable desymmetrization of the symmetric domain of the group $SL(2, Z)$.

By comparison with the transformations that define the BMG in Eq. (2.2), one learns that the action of the transformation \mathcal{C} in Eq. (2.2c) can be interpreted as a symmetry-quotienting mechanism induced on the desymmetrized domain (2.1) of the BMG. This symmetry-quotienting mechanism defines the language code of the BGM, such that periodic orbits of the BMG are classified according to this convention.

III. THE BIG BILLIARD GROUP

The big-billiard group (BBG) is obtained from the big billiard table, i.e. a domain defined by the three sides a , b , c ,

$$a : u = 0 \quad (3.1a)$$

$$b : u = -1 \quad (3.1b)$$

$$c : u^2 + u + v^2 = 0, \quad (3.1c)$$

for which bounces against the billiard sides are expressed by the following transformations on the UHP

$$Az = -\bar{z}, \quad (3.2a)$$

$$Bz = -\bar{z} - 2, \quad (3.2b)$$

$$Cz = -\frac{\bar{z}}{2\bar{z}+1}. \quad (3.2c)$$

The unquotiented big-billiard map \mathcal{T} consists of a suitable composition of the transformations (3.2); the sequence of this composition is obtained by the continued-fraction decomposition of the u^+ variable.

Periodic orbits of the big billiard are a phenomenon which is more complicated than its symmetry-quotiented versions [23]: there exists a particular Kasner transformation k_* (which depends on m and on the considered periodic orbit of \mathcal{T}) such that

$$\mathcal{T}^m(u^-, u^+) = k_*(u^-, u^+). \quad (3.3)$$

The set of six Kasner transformations is a realization of the S_3 permutation group (of order $3! = 6$). In fact, this permutation group consists of the identity, 3 transpositions [(12), (23) and (31)], and 2 cyclic transformations [(213) and (321)]. We recall that the *order* of a particular group element, such as k_* , is the smallest integer p such that $k_*^p = k_0$. As a transposition is of order 2, and a cyclic permutation, (123) or (321), of order 3, we see that the order p of k_* must be equal to $p = 1, 2$ or 3 . Therefore, by iterating (3.3), we get

$$\mathcal{T}^{mp}(u^-, u^+) = k_*^p(u^-, u^+) = (u^-, u^+), \quad (3.4)$$

and mp will be the smallest such integer. In other words, (u^-, u^+) is the initial point of a periodic orbit under the unquotiented billiard map \mathcal{T} , with period pm , where $p = 1, 2, 3$ is the order of k_* .

A. Cyclic identities

Given any x, y in A, B, C , periodic orbits are defined by imposing the condition (as alternatively solved in [46])

$$\prod_{n_i} T(x, y; n_i) z = z \quad (3.5)$$

where the epoch map $T(x, y; n_i)$ for a n_i -epoch BKL era of the xy type is defined as

- $T(x, y; n_i) = y(xy)^{\frac{n_i-1}{2}}, n_i$ odd,
- $T(x, y; n_i) = y(xy)^{\frac{n_i-1}{2}} x, n_i$ even,

with $\sum_i n_i = mp = n$.

IV. THE SMALL-BILLIARD GROUP

The small billiard is delimited by the sides G , B , R , defined as

$$G : u = 0, \quad (4.1a)$$

$$B : u = -\frac{1}{2}, \quad (4.1b)$$

$$R : u^2 + v^2 = 1. \quad (4.1c)$$

The transformation that describe the bounces of the billiard ball against the sides of the small billiard table (4.1) are those that define the $SL(2, Z)$ group, i.e.

$$R_1(z) = -\bar{z}, \quad (4.2a)$$

$$R_2(z) = -\bar{z} + 1, \quad (4.2b)$$

$$R_3(z) = \frac{1}{\bar{z}}, \quad (4.2c)$$

where no identification among the sides is present, and no symmetry-quotienting mechanisms for the R side of the small billiard table (4.1) is assumed.

The action of the transformations (4.2) on the oriented endpoints of the oriented geodesics that define the trajectories of the billiard ball on the small billiard table is obtained by imposing $v \equiv 0$ on the UPHP variable $z = u + iv$, and results in the small-billiard billiard map t , which acts diagonally on the reduced phase space variables $u \equiv u^\pm$ as

$$Gu = -u, \quad (4.3a)$$

$$Bu = -u - 1 \quad (4.3b)$$

$$Ru = \frac{1}{u}. \quad (4.3c)$$

Epochs on the small billiard table are defined as any trajectory joining any two walls of the small billiard sides; eras for the small billiard are defined as a succession of epochs starting from the side R .

c. The small-billiard map The CB-LKSKS map for the small billiard, $t_{CB-LKSKS}$, is defined by two different kinds of transformations, i.e.,

$$t^{1,2}z = T^{-1}SR_1T^{-n+1}z, \text{ for } (u^+, u^-) \in S_{ba}^1 \text{ and } (u^+, u^-) \in S_{ba}^2, \quad (4.4a)$$

$$t^{2',3,3'}z = T^{-1}SR_1T^{-n+1}R_3z, \text{ for } (u^+, u^-) \in S_{ba}^{2'}, (u^+, u^-) \in S_{ba}^3, \text{ and } (u^+, u^-) \in S_{ba}^{3'}. \quad (4.4b)$$

which act on the subregions of the reduced phase space S_{ba}^1 , S_{ba}^2 , S_{ba}^3 , $S_{ba}^{2'}$ and $S_{ba}^{3'}$ defined as

- S_{ba}^1 : $u^- < -\Phi$, $u^+ > u_\alpha(u^+)$, $-\Phi < u^- < -1$, $u^+ > u_\gamma(u^+)$;
- S_{ba}^2 : $-\Phi < u^- < -1$, $u_\alpha(u^+) < u^+ < u_\gamma(u^+)$;
- $S_{ba}^{2'}$: $u^- < -2$, $0 < u^+ < u_\alpha(u^+)$, $-2 < u^- < -\Phi$, $u_\gamma(u^+) < u^+ < u_\alpha(u^+)$;
- S_{ba}^3 : $-2 < u^- < -\Phi$, $u_\gamma(u^+) < u^+ < u_\beta(u^+)$, $-\Phi < u^- < -1$, $u_\alpha(u^+) < u^+ < u_\beta(u^+)$;
- $S_{ba}^{3'}$: $2 < u^- < -1$, $0 < u^+ < u_\beta(u^+)$;

where the functions

- $u_\alpha(u^+) : u^+ = -\frac{1}{u^-}$;
- $u_\beta(u^+) : u^+ = -\frac{u^-+2}{2u^-+1}$;
- $u_\gamma(u^+) : u^+ = -\frac{u^-+2}{u^-+1}$;

are defined. In particular, the function $u_\gamma(u^+)$ corresponds to the image of the function $u_\alpha(u^+)$ according to the transformation B .

d. *The language code for the small billiard group* The following language code for the small billiard is obtained

1. $R, B, G, BR, BG, RG, RB, GR, GB$;
2. RBR, BGR, RGB ;
3. \top_{XY^n} .

As one can straightforward verify, the sequence RGR is not allowed.

The trajectories \top_{XY^n} are a succession of epochs such that the first epoch starts form a R wall, and the last epoch ends on a R wall; between these two epochs, bounces between the G and the B wall take place, such that n trajectories are present.

As classified in [], the reduced phase space for the small billiard table is characterized curvilinear domains. In particular, is is possible to further divide these domain according to the preimage of the transformations (4.3) on the RG subdomain and on the RB with respect to the BR subdomain and to the GR one.

This way, the trajectories \top_{XY^n} are defined as a succession of transformations of the kind $\top_{XY^n} \equiv RXY...X'Y'$, where the transformations X, Y, X', Y' can be B or G . More in detail, they are defined on the small billiard reduced phase space as

1. $\top_{RG^n} \equiv RGB...BG, n \text{ odd}, u^+ > 1, u^- < -1/2, (u_\alpha(u^+), u_\beta(u^+), u_b^n(u^+), u_a^n(u^+))$;
2. $\top_{RG^n} \equiv RGB...GB, n \text{ even}, u^+ > 1, u^- < -1/2, (u_\alpha(u^+), u_\beta(u^+), u_b^{n-1}(u^+), u_a^{n+1}(u^+))$;
3. $\top_{RB^n} \equiv RBG...GB, n \text{ odd}, u^+ < 0, u^- > -1/2, (u_\alpha(u^+), u_\beta(u^+), U_b^n(u^+), U_a^n(u^+))$;
4. $\top_{RB^n} \equiv RBG...BG, n \text{ even}, u^+ < 0, u^- > -1/2, (u_\alpha(u^+), u_\beta(u^+), U_b^{n-1}(u^+), U_a^{n+1}(u^+))$;

The functions $u_b^n(u^+), u_a^n(u^+), U_b^n(u^+), U_a^n(u^+)$ are defined as

- $u_b^n(u^+) : u^+ = \frac{1}{2} \frac{-2nu^- + 2u^- + n^2 - 2n + 5}{-2u^- + n - 1}$;
- $u_a^n(u^+) : u^+ = \frac{1}{2} \frac{-4n + 7 - 2nu^- + 4u^- + n^2}{-2 - 2u^- + n}$;
- $U_b^n(u^+) : u^+ = -\frac{1}{2} \frac{2nu^- - 2u^- + n^2 - 2n + 5}{2u^- + n - 1}$;
- $U_a^n(u^+) : u^+ = -\frac{1}{2} \frac{3 + 2nu^- + 2n^2}{2u^- + n}$,

and correspond to the preimage of the RB and RG regions of the reduced phase space according to the pertinent combination of transformation G and B .

e. *Action of the SBG on the oriented endpoints.* Periodic orbits for the SBG are defined according by imposing that the oriented endpoints obey the condition

$$\top_{XY^n} u \equiv u. \quad (4.5)$$

According to the classification of the sequences \top_{XY^n} , the following quadratic equations with integer coefficients are obtained for the endpoints, respectively:

1. $\top_{RG^n}, n \text{ odd}, u^2 - mu + 1 = 0$, with $m \equiv \frac{n}{2} - 1$;
2. $\top_{RG^n}, n \text{ even}, u^2 - mu + 1 = 0$, with $m \equiv \frac{n}{2}$;
3. $\top_{RB^n}, n \text{ odd}, u^2 + mu + 1 = 0$, with $m \equiv \frac{n+1}{2}$;
4. $\top_{RB^n}, n \text{ even}, u^2 + mu - 1 = 0$, with $m \equiv \frac{n}{2}$.

These transformations are always hyperbolic, since their discriminants Δ_m reads

1. $\Delta_m \equiv \sqrt{m^2 + 4}$ for t_{RG^n} and t_{RB^n} , with n even;
2. $\Delta_m \equiv \sqrt{m^2 - 4}$ for t_{RG^n} and t_{RB^n} , with n odd,

as the minimum number of epochs in each \top_{XY^n} era is $n \leq 3$.

The periodic orbits defined by an even number n of epochs allow for a continued-fraction decomposition analogous to that obtained in the case of the Golden Ratio and of the 'silver ratios' for the big billiard. In the case of the big billiard, the phase-space points that define periodic orbits identified by these 'ratios' are placed along the function $u_\gamma(u^+)$ of the starting box F_{ba} .

In the case of an odd number of epochs, this decomposition does not hold any more. Furthermore, it is not possible to define any transformation able to map these trajectories to this kind of decomposition.

f. The epoch map for the unquotiented small billiard Collecting all the ingredients together, it is possible to generalize the content of the sequences \mathbb{T}_{XY^n} and to establish a map for the unquotiented variable $u \equiv u^\pm$ relating each first epoch of the small-billiard eras to each last epoch of the small-billiard eras, denoted by the phase-space points $u_F \equiv u_F^\pm$ and $u_L \equiv u_L^\pm$, respectively, as

$$u_L = \mathbb{T}_{GY^n} u_F \equiv (-1)^n (u - m), \quad (4.6)$$

and

$$u_L = \mathbb{T}_{BY^n} u_F \equiv (-1)^n (u + m), \quad (4.7)$$

with m defined in the above.

The era transition map is obtained by composing the epoch map with the transformation S , which accounts for the bounce of the R , side, such that the first epoch of the successive era is defined by the phase space points of the reduced phase space $u' \equiv u^{\pm'}$, i.e.

$$u' \equiv S \mathbb{T}_{XY^n}. \quad (4.8)$$

V. COMPARISON WITH THE BMG

As evident, there are several differences between the dynamics predicted by the action of the BMG, and that described by the SBG.

The main difference between the BMG and the SBG is the absence of symmetry-quotienting mechanisms. In fact, the transformation 2.2c of the BMG does not coincide with the transformation ?? of the SBG. The two groups are obtained via two different procedures: while the BMG is obtained from a desymmetrization of the domain of the $SL(2, Z)$ group, the BMG is due to the symmetry-quotienting of the big billiard table according to the presence of the symmetry walls.

As far as the *dynamical* properties of the small billiard are concerned, it is important to remark that, according to the desymmetrized shape of its domain, no identification is possible between the sides of the small billiard table, such that the different periodic phenomena described by the different t_{XY^n} maps cannot be identified.

On the other hand, as far as the *geometrical* properties of the small billiard are concerned, as already analyzed in [27], the (several) geometrical transformations that would allow one to recover squared subdomains for the reduced phase space of the SBG do not correspond to any succession of the language code for the SBG.

The BKL map for the small billiard consist of a different number of Weyl reflections, according to the different subregions of the reduced phase space where the first epoch of each small-billiard era is issued from. This property further explains the *physical* content of the \mathbb{T}_{XY^n} maps: as a different succession of matrices is implied for every \mathbb{T}_{XY^n} , in particular, those with n odd will contain an odd number of (Weyl) reflections, while those with n even will contain an even number of (Weyl) reflections.

A. Time evolution and cyclic identities

The Selberg trace formula will not therefore be described according to a suitable conjugacy subclass of the modular group, being $\Gamma_2(PGL(2, C))$ and $PGL(2, C)$ larger (with respect to the modular group) groups.

The cyclic identities of their language codes describing periodic orbits define the parity of the wavefunction solving the WDW equation for this implementation of the Einstein field equations, according to the number of BKL epochs contained in all the BKL eras which the periodic orbit consists of.

VI. QUANTUM REGIME

The semiclassical Poincaré return map [43], [45], is defined as

$$\tau_E \psi(q') = \int_{\Sigma} \tau_E(q', q) \psi(q') d^N q \quad (6.1)$$

where the integral is performed on a surface of section Σ , and is extended to the corresponding degrees of freedom of the phase space, with τ_E a billiard map obeying the consistency equation

$$0 = (1 + \tau_E)(1 - \tau_E)$$

and defines the Selberg ζ function according to the generalized [53] determinants $\det(1 - \tau_E)$ and $\det(1 + \tau_E)$ as

$$\zeta(s) = \det(1 + \tau_E)\det(1 - \tau_E) = \theta(s)\det(1 - \tau_E)$$

The expression of (6.1) for the case of cosmological billiards is then expressed according to the angle θ defined in Figure 2 as [5], or by its expression as a function of u^* , i.e. the value $u = \text{const}$ that defines a generic Poincaré section different from the billiard sides, and which parameterized the energy-shell reduced Liouville measure.

The classification of all the periodic orbits according to the different maps allows one to reconstruct the complete spectrum of the operator τ_E , which is the same as its self-adjoint operator U_E , which acts on the Maass waveforms.

The density levels dE of the quantum systems and their classical periodic orbits are related at the semiclassical

transition by the Selberg trace formula

$$dE = \bar{dE} + dE^{\text{osc}},$$

$$dE^{\text{osc}} = \sum_n A_n(E) e^{\frac{iS_n}{2\pi\hbar}},$$

being \bar{dE} and dE^{osc} the average (i.e. on containing singular orbits) contribution and the one corresponding to periodic orbits, respectively, with S_n expansion of the action and A_n the corresponding coefficient within the sum on periodic orbits n . The number of epochs in each BKL eras constituting periodic orbits for cosmological billiards define the parity (the sign) of the eigenvalues for the quantum-mechanical description of cosmological billiards on the UHP, corresponding to the BKL (towards the cosmological singularity) asymptotic limit of the WDW equation, solved by the (Maass) wavefunctions for cosmological billiards [27].

VII. OUTLOOK AND PERSPECTIVES

g. Outlook The mathematical analysis of the discretized dynamics of Cosmological Billiards has to be considered as relevant for the physical characterization of the quantum regime, the semiclassical transition and the classicalized states of a generic Universe with respect to the present observed Universe.

The definition of such characterization is needed for the comparison of the experimental evidence providing definition about the evolution of the Universe and the external (i.e. 'on the r.h.s. of the E.f.e.'s) which have to be supposed to have taken place as modifying the oscillatory behavior of a Generic Universe with respect to the present observed values of anisotropy, as well as for the anisotropy rates of the statistical distribution of matter densities as far as the investigation on Astrophysical scales is concerned: the introduction of isotropization mechanisms of chaotic models, and those of quasi-isotropization [51], can allow for a comparison with observational evidence, as well as the hypothesis of some inflation-generating mechanism [52].

The features of the spectrum of the energy levels of the wavefunction allow one to test the phenomenological effects obtained in Quantum-Gravity models [49], as far as the possible deformations of the background (geometrical) space are concerned, and allow one to compare the effectiveness of such Quantum-Gravity motivated investigations in modifying the chaotic properties of the billiard systems with the effects of classical scalar fields and vector fields [50]. The introduction of isotropization mechanisms of chaotic models, and those of quasi-isotropization [51], can allow for a comparison with observational evidence, as well as the hypothesis of some inflation-generating mechanism [52].

A complementary investigation line is constituted by [54] and [55].

h. Perspectives The analysis of the solution of the iterations of Cosmological Billiard Maps which define periodic orbits for cosmological billiards is fundamental for the comparison of the known results for the mathematical properties of periodic trajectories on domains on curved hyperbolic spaces, whose dynamics does not result as a symmetry-quotienting of a pre-existing dynamical system.

The implementation of quantum statistical maps for Cosmological Billiards is based on the comparison between the geometrical properties of these systems, which provide the analysis with a suitable group-theoretical structure, and the

symmetries of the dynamics, for which the symmetries of the metric tensor define a 'smaller' class of transformations, which characterizes the statistical description. This method of investigation, the Jacquet-Langlands correspondence, is framed in the broader Langland programme[47] [48]. Indeed, for cosmological billiards in $4 = 3 + 1$ spacetime dimensions, the semiclassical Poincaré return map defined by Eq. (6.1) is defined by a map τ_E for a given energy level corresponding to the classical configuration of energy E (at which the classical reduced phase space corresponds). More in detail, the operator τ_E corresponds to any of the classical billiard map defined for cosmological billiard, i.e. either the big billiard unquotiented map, or the big billiard Kasner quotiented maps, such as the BKL epoch map, the BKL era-transition map and the CB-LKSKS map, or the small billiard unquotiented map, or the small billiard BKL map.

The expression of (6.1) for the case of cosmological billiards is then expressed according to the angle θ defined in Figure ?? that defines a generic Poincaré section different from the billiard sides, and that parameterizes the energy-shell reduced Liouville measure.

The classification of all the periodic orbits according to the different maps allows one to reconstruct the complete spectrum of the operator τ_E .

Buondary conditions for cosmological billiards have already been thoroughly discussed in the literature. Both Neumann and Dirichlet boundary conditions have been proposed and motivated, according to different features that had to be described.

From (??), one learns that the two different conditions, i.e. $(1 + \tau_E)$ or $(1 - \tau_E)$ correspond to Neumann boundary conditions and to Dirichlet boundary conditions, which correpond, on their turn, to odd wavefunctions or to even wavefunctions.

It is crucial to remark that the identification of Eq. (6.1) to a semiclassical version of the BKL map operators for a fixed energy shell, and the identification of Eq. (??) with the choice of boundary conditions is restricted to either the surface of section Σ corresponds to one side of the billiard (for the purposes of this analysis, the side b of the big billiard), or the surface of section Σ does not correpond to a side of the billiard table, but the knowledge of both the unquotiented dynamics and the requested maps allows one to recast the proper geodesic flow through Σ , i.e. the one corresponding to that bouncing onto a side. The classical description of the cosmological billiards on the UPHP ant with its restricted phase space is obtained by fixing a given energy at which the Hamiltonian flow is calculated.

The operator τ_E defined in Eq. (6.1) can therefore be interpreted as the operator that, for each classical energy level E , acts on a semiclassical wavefunction (semiclassical in the sense that it is evaluated on a classical BKL configuration corresponding to a geiven sequence of epochs and eras).

At each energy level E , the operator(s) τ_E leave invariant (except for a \pm sign) the eigenfunctions of the quantum eigenvalue problem; these Selberg eigenfunctions (??) are defined as generalizing the Riemann ζ function to closed (i.e. periodic) orbits (instead of prime numbers).

The BKL map operators define the complete set of periodic orbits of cosmological billiards (from a classical point of view).

The corresponding system of operators τ_E can therefore be interpreted as *the semiclassical operator which extracts an eigenvalue for each closed geodesics, which correspond to periodic orbit, according to it content of BKL epochs, BKL eras and the chosen symmetry-quotienting mechanism, and whose complete spectra are now classified.*

VIII. CONCLUDING REMARKS

The aim of this investiagtion has been to define periodic phenomena for Cosmological Billiards in $4 = 3 + 1$ spacetime dimensions. In particular, quantum BKL maps and the semiclassical BKL maps have eigenvalues which define the sign acquired by the wavefunctions for cosmological billiards, whose parity is defined by the corresponding periodicity phenomena.

The paper is organized as follows.

In the Introduction (I), Cosmological Billiards are introduced.

In Section II, the Billiard Modular Group is defined.

The Big Billiard Group is defined in Section III, and the Small Billiard in Section IV, for which the differences with the Billiard Modular Group are outlined in Section V as those characterizing the symmetries of the solutions to the Einstein field equations.

For this billiard groups, the language code is stated, and periodic orbits are spelled out and analyzed.

The sign acquired by the Maass wavefunctions under the quantum BKL maps ?? characterizes the Selberg trace formula for Cosmological Billiards, which also defines the energy-level densities according to the quantum BKL maps

for periodic orbits, as analyzed in Section VI. As analyzed in Section VI and initially commented in Section VII, the quantum BKL map operators are the same self-adjoint operator of the quantum BKL operators defined on the solution of the minisuperspace reduction of the WDW equation for cosmological billiards. This result strictly applies to the analysis of [48] and [53] and further specifies the analysis of [56] and [43] for Cosmological Billiards,

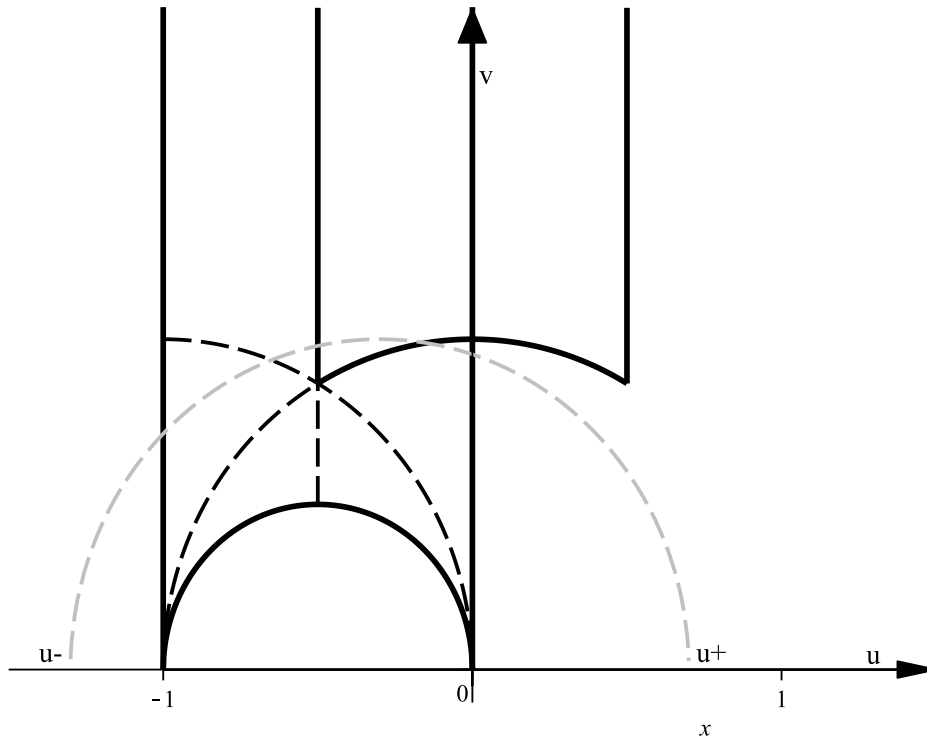


FIG. 1. The domains of the big billiard $\Gamma_2(PGL(2, C))$, the small billiard, the $SL(2, Z)$ group, the $SL(2, C)$ group and the billiard modular group. In particular, the domain of the big billiard is delimited by the geodesics $u = 0$, $u = -1$, $u^2 + v^2 = 1$; the domain of the small billiard, which coincides with that of the $SL(2, Z)$ group, is delimited by the geodesics $u = 0$, $u = -1/2$ and $u^2 + v^2 = 1$; the domain of the $SL(2, C)$ group is delimited by the geodesics $u = 1/2$, $u = -1/2$ and $u^2 + v^2 = 1$; the domain of the billiard modular group is delimited by the geodesics $u = 0$, $u = 1/2$ and $u^2 + v^2 = 1$. They are all plotted by solid black lines. The symmetry lines of the big billiard are represented by the dashed black lines. An oriented geodesic is drawn as gray dashed circle, and the oriented endpoints u^+ and u^- are indicated on the u axes.

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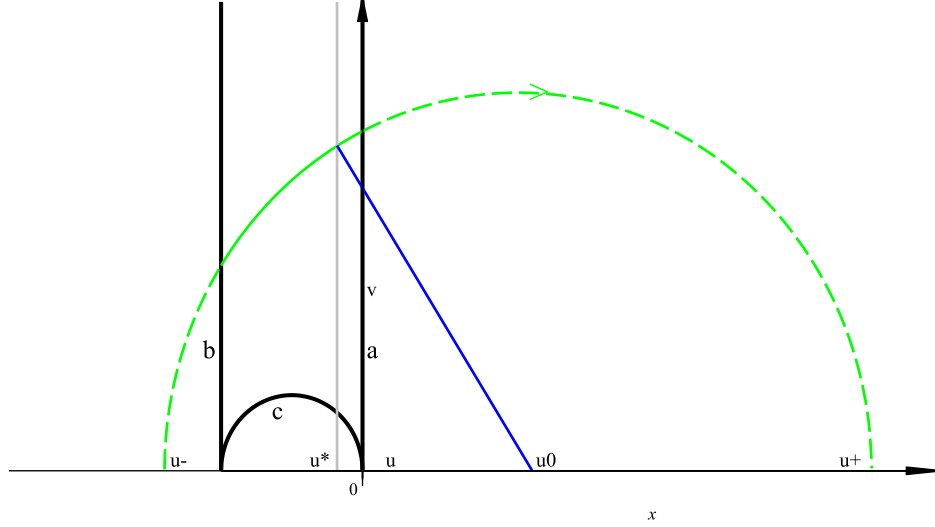


FIG. 2. The angle θ is comprehended within the axis of abscissae and the radius r of the geodesics connecting the center of the geodesics u_0 with the intersection point between the generalized Poincaré surface of section $u_* = \text{const}$ and the same geodesics, such that $\cos \theta = (u_0 - u^*)/r$.

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